FEM PROGRAM FOR A COUPLING ANALYSIS OF A HYPERELASTIC SOLID AND STATIC LIQUID TO SIMULATE THE RETINAL DETACHMENT OPERATION ON AN EYEBALL

眼球の網膜剥離手術シミュレーションのための超弾性体と静止液体の連成解析FEMプログラム

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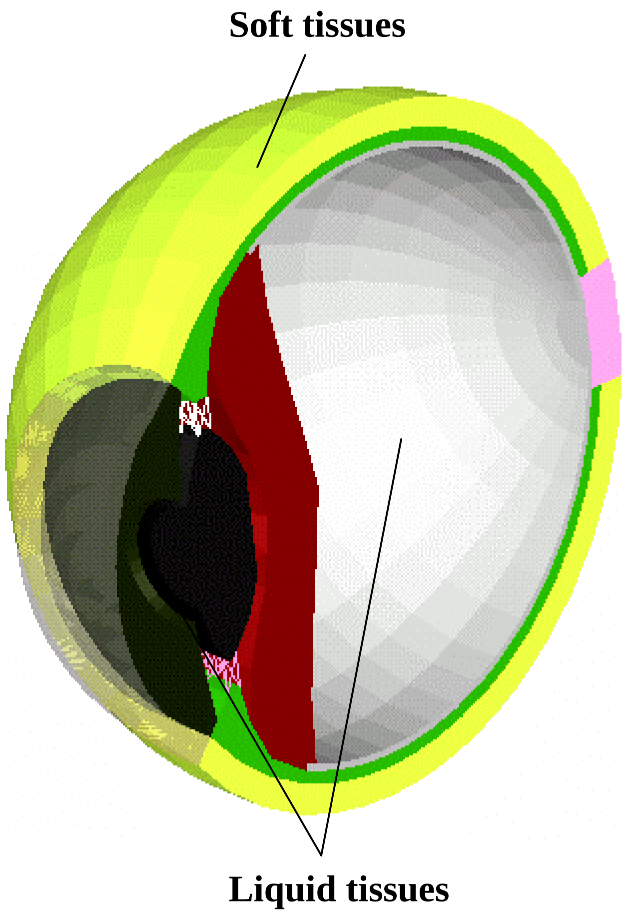
Structure of an eyeball



Features of program

- **Method:**

  Mixed finite element method (Lagrange multiplier method).

- **Element types:**

  4/1, 8/1, 9/3.

- **Boundary conditions:**

  Displacement, Force, Pressure.

- **DEVELOPMENT OF THE 2-D INCOMPRESSIBLE HYPER-ELASTIC PROGRAM (2次元非圧縮性超弾性プログラムの開発)**
  - Features of program
  - FEM formulation for the incompressible hyperelastic material ppppanalysis
  - Numerical experiments

- **DEVELOPMENT OF THE 2-D SOLID-LIQUID COUPLING ANALYSIS PROGRAM (2次元固体ー液体連成解析プログラムの開発)**
  - FEM formulation for the solid-liquid coupling analysis
  - Numerical experiments

FEM formulation for the incompressible hyperelastic material analysis

Strain energy function:

$$W = W(I_1, I_2) \tag{1}$$

Incompressibility constraint condition:

$$J = 1 \quad or \quad I_3 = 1 \tag{2}$$

J : determinant of the Jacobian matrix.

I_1 , I_2 , I_3 : invariants of the right Cauchy-Green deformation tensor.

Reduced invariants:

$$\bar{I}_1 = I_1 J^{-2/3} \qquad \bar{I}_2 = I_2 J^{-4/3} \tag{3}$$

Modified Strain energy function:

$$W = W(\bar{I}_1, \bar{I}_2) \tag{4}$$

FEM formulation for the incompressible hyperelastic material analysis

Potential energy functional:

$$\Phi = \int_{V0} (W(\bar{I}_1, \bar{I}_2) + 2\lambda(J-1))\,dV - g(u) \qquad (7)$$

λ : Lagrange multiplier.

$g(u)$:Potential energy of the external forces.

Variational equation:

$$\begin{aligned} \delta\Phi &= \int_{V0} (\partial W / \partial\varepsilon_{ij} + 2\lambda(\partial J / \partial\varepsilon_{ij}))\delta\varepsilon_{ij}\,dV \\ &+ \int_{V0} 2(J-1)\delta\lambda\,dV - g(\delta u) \\ &= 0 \end{aligned} \qquad (8)$$

Finite element discretization:

$$\begin{cases} \int_{V0} (\partial W / \partial\varepsilon_{ij} + 2\lambda(\partial J / \partial\varepsilon_{ij}))((\partial\phi_M / \partial X_j)u_{Mn} + \delta_{jn})(\partial\phi_N / \partial X_i)dV = r_{Nn} \\ \int_{V0} \varphi_R\, 2(J-1)dV = 0 \end{cases} \qquad (10)$$

FEM formulation for the incompressible hyperelastic material analysis

Element stiffness equation:

$$\begin{bmatrix} k_1^{i-1} & k_2^{i-1} \\ k_3^{i-1} & 0 \end{bmatrix} \begin{bmatrix} \Delta u^i \\ \Delta\lambda^i \end{bmatrix} = \begin{bmatrix} fu_1^{i-1} \\ f\lambda_2^{i-1} \end{bmatrix} + \begin{bmatrix} r \\ 0 \end{bmatrix} \quad (11)$$

Global stiffness equation:

$$\begin{bmatrix} K_1^{i-1} & K_2^{i-1} \\ K_3^{i-1} & 0 \end{bmatrix} \begin{bmatrix} \Delta U^i \\ \Delta\lambda^i \end{bmatrix} = \begin{bmatrix} FU_1^{i-1} \\ F\lambda_2^{i-1} \end{bmatrix} + \begin{bmatrix} R \\ 0 \end{bmatrix} \quad (12)$$

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$$\left[\begin{array}{l} U^i = U^{i-1} + \omega\,\Delta U^i \\ \lambda^i = \lambda^{i-1} + \omega\,\Delta\lambda^i \end{array} \right. \quad (13)$$

ω: Relaxation factor.

Numerical experiments

Table 1 Calculation times and numbers of iteration steps

	Calculation time (sec.)			Number of iteration step		
	4/1	8/1	9/3	4/1	8/1	9/3
Model 1	30	54	72	94	123	113
Model 2	144	246	312	319	429	371
Model 3	42	59	66	60	72	61
Model 4	144	214	246	110	121	116

FEM formulation for the solid-liquid coupling analysis

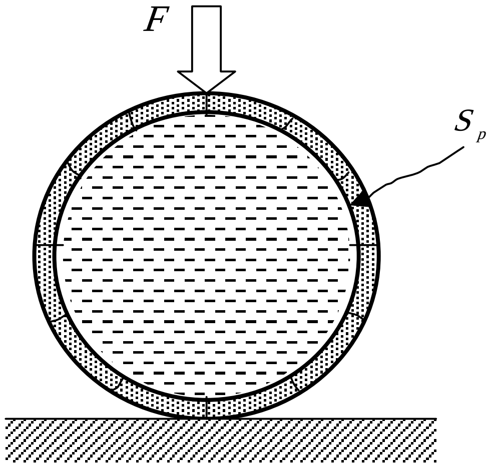

Fig. 3 Circular vessel filled with liquid under a diametrical load

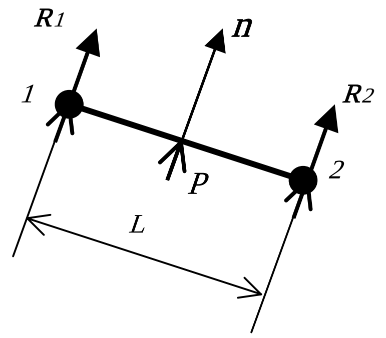

Fig. 4 A segment of the internal surface of the vessel

Nodal forces equivalent to the liquid pressure:

$$\boldsymbol{R}_1 = \boldsymbol{R}_2 = \frac{1}{2} \bullet L(u_e) \bullet P(u_{S_p}) \bullet \boldsymbol{n}(u_e)$$

L: Segment length, P: Liquid pressure, $\boldsymbol{n}$: inward normal to the segment, u_e: Node displacements of the segment, u_{S_p}: Displacements of the all nodes on the internal surface of the vessel.

FEM formulation for the solid-liquid coupling analysis

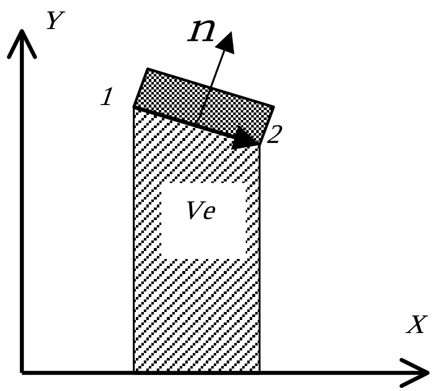

Fig. 5 Calculation of the liquid volume

Relation between pressure change and volume change in the case of a static liquid:

$$\Delta P = -K \frac{\Delta V}{V_0} \tag{15}$$

K : Liquid bulk modulus, V_0 : Initial liquid volume.

Volume change:

$$V(u_{s_p}) = \sum V_e(u_e) \tag{16}$$

$$V_e(u_e) = \frac{1}{2}(x_2 - x_1)(y_1 + y_2) \tag{17}$$

$$\Delta V(u_{s_p}) = V(u_{s_p}) - V_0 \tag{18}$$

Pressure change:

$$\Delta P(u_{s_p}) = -K\left(\frac{V(u_{s_p})}{V_0} - 1\right) \tag{19}$$

Liquid pressure:

$$P(u_{s_p}) = P_0 - K\left(\frac{V(u_{s_p})}{V_0} - 1\right) \tag{20}$$

FEM formulation for the solid-liquid coupling analysis

Equivalent nodal forces:

$$\boldsymbol{R}_1 = \boldsymbol{R}_2 = \boldsymbol{R}(u_{S_p}) \tag{21}$$

$$\begin{cases} \int_{V0} (\partial W / \partial \varepsilon_{ij} + 2\lambda(\partial J / \partial \varepsilon_{ij}))((\partial \phi_M / \partial X_j) u_{Mn} + \delta_{jn})(\partial \phi_N / \partial X_i) dV = r_{Nn} + r^L_{Nn}(u_{S_p}) \\ \int_{V0} \varphi_R \, 2(J-1) dV = 0 \end{cases} \tag{22}$$

Global stiffness equation for the solid-liquid coupling analysis:

$$\begin{bmatrix} \boldsymbol{K}_1^{i\text{-}1} + \boldsymbol{K}_{1L}^{i-1} & \boldsymbol{K}_2^{i\text{-}1} \\ \boldsymbol{K}_3^{i\text{-}1} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{U}^i \\ \Delta \boldsymbol{\lambda}^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{FU}_1^{i\text{-}1} \\ \boldsymbol{F\lambda}_2^{i\text{-}1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{R} + \boldsymbol{R}_L^{i\text{-}1} \\ \boldsymbol{0} \end{bmatrix} \tag{23}$$

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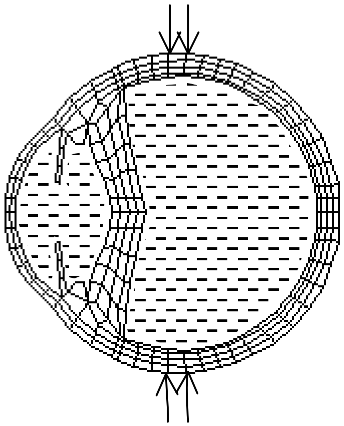
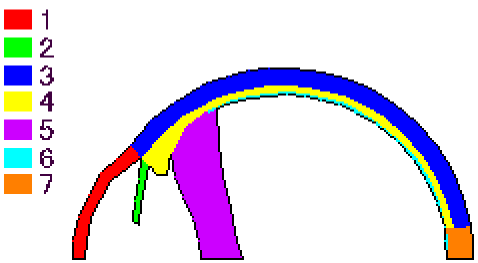
Solution: $\boldsymbol{U}$, $\boldsymbol{\lambda}$

Numerical experiments



(a) Finite element division



(b) Tissue division

Fig. 8 Analytical model for the buckling operation on an eyeball

Soft tissues : neo-Hooke material, $W = c_1(I_1 - 3)$

Table 1 Material constants for the tissues

	1- cornea	2-iris	3-sclera	4-choroid	5-vitreous	6-retina	7-optic nerve
C_1	0.033	0.0083	0.083	0.0083	0.00002	0.00083	0.0083

Liquid : bulk modulus 2083.3 *MPa*

CONCLUSIONS

- A two-dimensional incompressible hyperelastic FEM program that employs the mixed FEM formulation was developed to suit the soft tissues of the eyeball. Three types of mixed element, i.e., 4/1, 8/1 and 9/3, were introduced into this program and the boundary conditions prescribed by the displacement, force and pressure were made applicable. This program was used to perform several numerical experiments. The results show that this program was effective, and that the 4/1 element gave the best performance among the three element types in respect of both the accuracy and the efficiency of the analysis.

- Based on this program, a two-dimensional program for a coupling analysis of a hyperelastic solid and static liquid was then developed. Two numerical tests, including an analysis of the buckling operation on an eyeball, were carried out. The results of these tests were satisfactory, and in particular, the result for the analysis of the buckling operation qualitatively demonstrated that the initial internal pressure in the eyeball is one factor that can influence the effectiveness of the operation.

- We plan to continue this work and develop a three-dimensional coupling analysis program by expanding the two-dimensional program developed here in the future.